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Indecomposable continua and the Julia sets of polynomials, II

Douglas K. Childers [a], John C. Mayer [a,*], James T. Rogers Jr [b]

[a] *Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170, USA*
[b] *Department of Mathematics, Tulane University, New Orleans, LA 70118-5698, USA*

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Abstract

We find necessary and sufficient conditions for the connected Julia set of a polynomial of degree $d \geqslant 2$ to be an indecomposable continuum. One necessary and sufficient condition is that the impression of some prime end (external ray) of the unbounded complementary domain of the Julia set J has nonempty interior in J. Another is that every prime end has as its impression the entire Julia set. The latter answers a question posed in 1993 by the second two authors.

We show by example that, contrary to the case for a polynomial Julia set, the image of an indecomposable subcontinuum of the Julia set of a rational function need not be indecomposable.

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1. Introduction

Let $\mathbb{C}$ denote the complex plane and let $\mathbb{C}_\infty$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ denote a polynomial of degree $d \geqslant 2$ and $J = J(f)$ the Julia set of f.

* Corresponding author.
E-mail addresses: childers@math.uab.edu (D.K. Childers), mayer@math.uab.edu (J.C. Mayer), jim@math.tulane.edu (J.T. Rogers).

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Recall that J is fully invariant under f, i.e., $J = f(J) = f^{-1}(J)$. By K we denote the *filled Julia set* consisting of J together with its bounded complementary domains in $\mathbb{C}$, and we let $U_\infty = \mathbb{C}_\infty \setminus K$. Note that $J = \partial U_\infty = \partial K$. We suppose all the critical points of f have bounded orbits, in which case J is connected [10, Theorem 9.5].

A *continuum* is a compact, connected metric space. A continuum is *decomposable* if it can be written as the union of two of its proper subcontinua; otherwise, it is *indecomposable*.

A longstanding question in holomorphic dynamics asks if the Julia set of a rational function can be an indecomposable continuum. The last two authors attacked this question in 1993 and gave several necessary and sufficient conditions in [9, Theorem 3.2] for the Julia set of a polynomial to be an indecomposable continuum. In Theorem 3.4 of the same paper, these authors obtained even better results on this problem, but this theorem was proved only for quadratic polynomials. At the time these authors suggested (see the paragraph preceding Theorem 3.4 of [9]) that these "better results" ought to be true for any polynomial.

The purpose of this paper is to prove these "better results" for all polynomials. Along the way we answer questions, simplify proofs, and extend results of the previous paper [9]. To state our results we must remind the reader of a few technical terms from prime end theory and complex dynamics. A convenient source for definitions and details about Julia sets, prime ends, and the applicability of prime ends to Julia sets is [10]. General references for prime end theory include [3,17,13,2].

By $\mathbb{D}$ we denote the open unit disk in $\mathbb{C}_\infty$ and we let $\mathbb{D}_\infty = \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ be the "unit disk" about ∞. Let $\phi : \mathbb{D}_\infty \to U_\infty$ be the Bötkher uniformization of U_∞, a conformal isomorphism that conjugates f on U_∞ to $z \to z^d$ on $\mathbb{D}_\infty$ [10, Theorem 9.5]. By $[\infty, \eta)$ we denote a radial ray in $\mathbb{D}_\infty$ from ∞ to $\eta \in \partial\mathbb{D}_\infty$, and we parameterize $\partial\mathbb{D}_\infty$ as $\mathbb{R}/\mathbb{Z}$. Thus, $z \to z^d$ on $\partial\mathbb{D}_\infty$ becomes $\eta \to d\eta \pmod 1$. In the literature of complex dynamics, the image $R_\eta = \phi([\infty, \eta))$ in U_∞ is called the *external ray* at η. Each external ray uniquely corresponds to a prime end of U_∞, so we will refer to prime ends by the corresponding $\eta \in \partial\mathbb{D}_\infty$.

The *principal continuum* of the prime end η, denoted $\mathrm{Pr}(\eta)$, is the remainder $\overline{R_\eta} \setminus R_\eta$. The *impression* of the prime end η, denoted $\mathrm{Im}(\eta)$, is the continuum

$$\mathrm{Im}(\eta) = \{ w \in \mathbb{C}_\infty \mid \exists \{z_i\} \subset \mathbb{D}_\infty,\ z_i \to \eta \text{ and } \phi(z_i) \to w \}.$$

Note that $\emptyset \neq \mathrm{Pr}(\eta) \subset \mathrm{Im}(\eta)$; furthermore, $\mathrm{Pr}(\eta)$ may be nondegenerate, and the containment may be proper. However, Fatou showed that the prime ends for which $\mathrm{Pr}(\eta)$ is degenerate is a set of full measure in $\partial\mathbb{D}_\infty$ [10, Theorem 17.4], and Collingwood showed that the prime ends for which $\mathrm{Im}(\eta) = \mathrm{Pr}(\eta)$ form a residual set (i.e., containing a dense G_δ) in $\partial\mathbb{D}_\infty$ [3,13]. Topologists call a prime end η for which $\mathrm{Pr}(\eta) = J$ a *simple dense canal* or *Lake-of-Wada channel*.

Using the notation above, we can now state our main theorem.

Theorem 1.1. *Suppose J is the connected Julia set of the polynomial f. Then the following are equivalent*:

(a) *J is an indecomposable continuum.*
(b) *For some prime end η of U_∞,* $\mathrm{Im}(\eta)$ *has nonempty interior in J.*
(c) *For every prime end η of U_∞,* $\mathrm{Im}(\eta) = J$.

Theorem 1.1 generalizes Theorem 3.4, and expands Theorem 3.2, of the previous paper [9]. Previously, we could only prove (b) $\Rightarrow$ (c) for quadratic polynomials.

The following are useful facts about indecomposable continua which we will use subsequently in our proof of Theorem 1.1.

(1) Each proper subcontinuum of an indecomposable continuum X has empty interior in X [6, Theorem 3-41].
(2) If a nonseparating plane continuum X contains in its boundary an indecomposable subcontinuum Y, then for some prime end η, $\mathrm{Im}(\eta) \supset Y$ [16, Theorem 4].
(3) If a plane continuum X has in its complement a Lake-of-Wada channel (that is, for some prime end η, $\mathrm{Pr}(\eta) = X$), then X is indecomposable [16, Theorem 3].
(4) If a nonseparating plane continuum X has in its complement a prime end η for which $\mathrm{Im}(\eta) = \partial X$, then either ∂X is indecomposable or ∂X is the union of two proper indecomposable subcontinua [16, Theorem 2].
(5) Let X be a nonseparating plane continuum, and Y an indecomposable subcontinuum of ∂X such that Y has nonempty interior in ∂X. If Z is a subcontinuum of ∂X with nonempty interior in ∂X such that $\mathrm{Int}_{\partial X}(Z \cap Y) \neq \emptyset$, then $Z \supset Y$ ([5, Theorem 1]; compare [8, Theorem 2.3]).

A *composant* $C(x)$ of an indecomposable continuum X is the union of all proper subcontinua containing the point $x \in X$. An indecomposable continuum X is the union of uncountably many pairwise disjoint composants, each dense in X [6, Theorems 3-44 through 3-47]. The following theorem generalizes Corollary 3.5 of [9].

Theorem 1.2. *Suppose the Julia set J of the polynomial f is an indecomposable continuum. Then no composant of J can contain the principal continuum of more than one prime end of U_∞. Moreover, prime ends corresponding to Lake-of-Wada channels of U_∞ form a residual set in $\partial\mathbb{D}_\infty$.*

Theorem 1.2 indicates just how complicated an indecomposable polynomial Julia set would be. In particular, everywhere you looked (from the outside) there would be the mouth of a Lake-of-Wada channel, and nowhere could you approach a point, or a proper subcontinuum, along more than one external ray. In particular, no two landing external rays could land in the same proper subcontinuum of J.

2. Proofs of Theorems 1.1 and 1.2

Principal continua and impressions behave very nicely with respect to the Bötkher uniformization. The following is a special case of [9, Lemma 2.1].

Lemma 2.1. *For each prime end η of U_∞, $f(\mathrm{Im}(\eta)) = \mathrm{Im}(d\eta)$ and $f(\mathrm{Pr}(\eta)) = \mathrm{Pr}(d\eta)$.*

We now prove Theorem 1.1. Consider the conditions below. We will show that they are equivalent.

(1) J is an indecomposable continuum.
(2) Some indecomposable subcontinuum of J has nonempty interior in J.
(3) For some prime end η of U_∞, $\mathrm{Im}(\eta)$ has nonempty interior in J.
(4) For some prime end η of U_∞, $\mathrm{Im}(\eta) = J$.
(5) For a dense subset D of $\partial\mathbb{D}_\infty$, for each $\eta \in D$, $\mathrm{Im}(\eta) = J$.
(6) For every prime end η of U_∞, $\mathrm{Im}(\eta) = J$.
(7) The set of Lake-of-Wada channels of U_∞ is a residual set in $\partial\mathbb{D}_\infty$.
(8) Some prime end η of U_∞ is a Lake-of-Wada channel.

We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) first. Then we show that (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (1) These steps are already found in our paper [9], so we only give a brief indication here. Finally, we show that (4) $\Rightarrow$ (5), the step previously done only for degree 2. In a simplification of our previous argument, we do not need to show first that (4) $\Rightarrow$ (1).

The reader is referred to [9, Theorem 3.2] for the proof of (1) $\Rightarrow$ (2) (trivial), (2) $\Rightarrow$ (3) (fact 2 above), and (3) $\Rightarrow$ (4) ([10, Corollary 14.2] and Lemma 2.1). The reader is referred to [9, Theorem 3.4] for the proofs of (5) $\Rightarrow$ (6) (a prime end crosscut argument using the dense set D), (6) $\Rightarrow$ (7) (since prime ends for which $\mathrm{Im}(\eta) = \mathrm{Pr}(\eta)$ are residual), (7) $\Rightarrow$ (8) (trivial), and (8) $\Rightarrow$ (1) (fact 3 above).

The key to the proof of (4) $\Rightarrow$ (5) is Theorem 2.2, below, an improvement of Theorem 4.5 in Section 4 of [15]. We provide a proof of Theorem 2.2 herein for completeness. Theorem 2.2 answers Question 3.8 of [9] in the affirmative. We then show how it allows us to prove step (4) $\Rightarrow$ (5) of the proof of Theorem 1.1, answering Question 3.7 of [9] in the affirmative.

Theorem 2.2. *Let $g : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational function, and let X be a compact subset of $\mathbb{C}_\infty$. If Y is a compact subset of X with empty interior in X, then $g(Y)$ has empty interior in $g(X)$.*

Proof. By way of contradiction, assume $g(Y)$ has nonempty interior in $g(X)$. As a notational convenience, for $A \subset \mathbb{C}_\infty$, we use $\widetilde{A}$ to denote $g(A)$. Let d be the degree of g. For each positive integer m, define the set

$$Q_m = \{ y \in Y \mid g^{-1}(\tilde{y}) \text{ contains at least } m \text{ points of } Y \}.$$

Note that $Q_1 = Y$ and that $Q_m = \emptyset$ for all $m > d$. Hence, there exists a largest integer n satisfying the condition

$$Q_n \cup \{ y \in Y \mid \tilde{y} \text{ is a critical value of } g \} = Y.$$

We establish a contradiction by proving that

$$Q_{n+1} \cup \{ y \in Y \mid \tilde{y} \text{ is a critical value of } g \} = Y.$$

Let y be a point of Y such that $\tilde{y}$ is not a critical value and $\tilde{y}$ is in the interior of $\widetilde{Y}$ with respect to $\widetilde{X}$. Since Y has no interior in X, it follows that each point of Y is the limit of a sequence of points of $X \setminus Y$. Let (x_i) be a sequence of points of $X \setminus Y$ that converges to y. Then $(\tilde{x}_i)$ converges to $\tilde{y}$. Without loss of generality, no $\tilde{x}_i$ is a critical value. Moreover, since $\tilde{y}$ is in the interior of $\widetilde{Y}$ with respect to $\widetilde{X}$, we may assume that $\tilde{x}_i \in \widetilde{Y}$ for all i. Since $y \in Q_n$, we may assume that for each i, $g^{-1}(\tilde{x}_i)$ contains at least n points of Y. Call them $y_i^1, \dots, y_i^n$. By choosing a subsequence, we may assume (y_i^1) converges to a point y^1 in Y. Taking more subsequences, if necessary, we find points $y^1, y^2, \dots, y^n$ of Y belonging to $g^{-1}(\tilde{y})$ such that (y_i^j) converges to y^j, for $1 \leqslant j \leqslant n$. Each of these points maps to $\tilde{y}$. Since $\tilde{y}$ is not a critical value, none of these points is a critical point. Hence, all of them are distinct from each other and from y. Therefore, the point y is in Q_{n+1}. Since y was arbitrary, we have the desired contradiction. □

We now complete the proof of Theorem 1.1 by showing that (4) ⇒ (5). Suppose $\eta_0 \in \partial\mathbb{D}_\infty$ is a prime end such that $\mathrm{Im}(\eta_0) = J$. By fact 4, there are two cases: either J is indecomposable, or $J = A \cup B$, where each of A and B are proper indecomposable subcontinua of J.

Define the set

$$D = \{\eta \in \partial\mathbb{D}_\infty \mid \exists n \in \mathbb{Z}^+,\ d^n\eta = \eta_0\}.$$

Since D is the full inverse orbit of η_0 under $\eta \to d\eta \pmod 1$, D is dense in $\partial\mathbb{D}_\infty$. If for every $\eta \in D$, $\mathrm{Im}(\eta) = J$, then we are done. So we may assume there exists a $k > 0$ and $\eta_1 \in D$ such that $d^k\eta_1 = \eta_0$ and $\mathrm{Im}(\eta_1) \neq J$.

It follows from Theorem 2.2 that for every $\eta \in D$, $\mathrm{Im}(\eta)$ has interior in J. If J is indecomposable, then by fact 1, being a proper subcontinuum, $\mathrm{Im}(\eta_1)$ has no interior in J, a contradiction. So we may assume that $J = A \cup B$. Since A and B are closed in J, $A \setminus B$ and $B \setminus A$ are open in J. Hence, by [10, Corollary 14.2], there is an n such that $f^n(A) = J = f^n(B)$. Without loss of generality, since $J(f^n) = J(f)$ [10, Lemma 4.2] we may assume $n = 1$.

Let $\eta_2 \in D$ such that $d\eta_2 = \eta_1$. Since $\mathrm{Im}(\eta_2)$ has nonempty interior in $J = A \cup B$, it follows that $\mathrm{Int}_J(\mathrm{Im}(\eta_2) \cap A) \neq \emptyset$ or $\mathrm{Int}_J(\mathrm{Im}(\eta_2) \cap B) \neq \emptyset$. Since A and B are indecomposable, it follows by fact 5 above that $\mathrm{Im}(\eta_2) \supset A$ or $\mathrm{Im}(\eta_2) \supset B$. Without loss of generality, $\mathrm{Im}(\eta_2) \supset A$. Then by Lemma 2.1, $\mathrm{Im}(\eta_1) = \mathrm{Im}(d\eta_2) = f(\mathrm{Im}(\eta_2)) \supset f(A) = J$. This contradicts $\mathrm{Im}(\eta_1) \neq J$.

This completes the proof of (4) ⇒ (5), and so of Theorem 1.1.

The proof of the first part of Theorem 1.2 is like that of Proposition 11 in [14] or Theorem 3.3 in [2]. The idea is that an indecomposable Julia set cannot contain a cut point or "cut continuum". The proof of the second part is (6) ⇒ (7) in the proof of Theorem 1.1.

3. Indecomposable subcontinua of Julia sets

In [15], the third author showed that indecomposability of subcontinua of a polynomial Julia set is "preserved" under the polynomial map.

Theorem 3.1. (Rogers) *If f is a complex polynomial and X is an indecomposable subcontinuum of the Julia set J of f, then $f(X)$ is also an indecomposable continuum.*

The purpose of the example below is to show that Theorem 3.1 cannot be extended to the Julia set of a rational function. We first need some topological preliminaries about the Sierpinski universal plane curve.

3.1. Sierpinski universal plane curve

The Sierpinski universal curve Σ [12, p. 9], also known as the Sierpinski carpet [4], has the property that any one dimensional plane continuum can be embedded in Σ. A continuum with such a property is said to be universal. The Sierpinski universal curve can be considered as a subcontinuum of $\mathbb{C}$. We will use the term Sierpinski curve to mean any subcontinuum of $\mathbb{C}$ homeomorphic to Σ. Clearly any Sierpinski curve is universal.

Milnor and Tan Lei [11] were the first to publish a proof showing the existence of a rational map with its Julia set being a Sierpinski curve. More recently Devaney and his co-authors in [1] have found a class of rational maps, each one having its critical points contained in the basin of attraction at infinity, and such that its Julia set is a Sierpinski curve. This construction is also described in [4]. We will use properties of Sierpinski curves to find an indecomposable subcontinuum Y of J such that $f(Y)$ is decomposable, where J is the Julia set of an element from the class described in [1,4].

It seems appropriate to restrict ourselves to the topological space $\mathbb{C}$. Hence we will only consider subcontinua of the complex plane. The following characterization of Sierpinski curves is due to Whyburn [18].

Theorem 3.2. *A plane continuum S is a Sierpinski curve iff S is a locally connected, nowhere dense subcontinuum for which the boundaries of complementary domains are pairwise disjoint simple closed curves.*

For a Sierpinski curve S we will use *outer boundary* to mean the boundary of the unbounded complementary domain of S. In proving Theorem 3.2, Whyburn [18] establishes the following properties.

Proposition 3.3. *Let S_0 and S_1 be Sierpinski curves with C_0 and C_1 being the corresponding outer boundaries. Then any homeomorphism $h : C_0 \to C_1$ can be extended to a homeomorphism $H : S_0 \to S_1$.*

Proposition 3.4. *Let S be a Sierpinski curve and let G be a simply connected domain whose boundary $C = \partial G$ forms a simple closed curve in S. Suppose that for every complementary domain $D \subset \mathbb{C} \setminus S$, if $\partial D \subset \overline{G}$ then $\partial D \cap C = \emptyset$. Then $T = \overline{G} \cap S$ is a Sierpinski curve with outer boundary C.*

From Theorem 3.2 it is clear that a Sierpinski curve contains points that are not in the boundary of any complementary domain. These points are often referred to as *irrational points* of a continuum. However in dynamics, when the Julia set is connected, these points are more commonly called *buried points*. Given the subject of our paper it seems more appropriate to use the latter terminology.

Definition 3.5. For a continuum K define its buried points B_K to be the set (possibly empty) of all points in K that are not lying in the boundary of any complementary domain. More specifically, if $\mathcal{D}$ denotes the collection of components of $\mathbb{C} \setminus K$ then $B_K = \mathbb{C} \setminus \bigcup_{D \in \mathcal{D}} \overline{D}$.

As noted earlier, every Sierpinski curve must contain buried points. It is well known (and follows from Proposition 3.6 below) that the set of buried points in a Sierpinski curve forms an arcwise connected subset. The next proposition, as noted by Whyburn [18], follows from a result of R.L. Moore and plays a crucial role in our investigation.

Proposition 3.6. *Let S be a Sierpinski curve, let $\mathcal{D}_0$ be the collection of all bounded complementary domains, and let D_∞ be the unbounded complementary domain. Let $\widetilde{S}$ be the decomposition space of S formed by the partition $\{\partial D\colon\ D \in \mathcal{D}_0\} \cup \{x\colon\ x \in B_S \cup \partial D_\infty\}$. Then $\widetilde{S}$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C}\colon\ |z| \leqslant 1\}$.*

The following theorem and proof is a generalization of Krasinkiewicz [7]. In [7], he proved that any homeomorphism between two Sierpinski curves must take the boundaries of complementary domains to boundaries of complementary domains. He also showed that given any two buried points in a Sierpinski curve S, there is a homeomorphism of S that carries one to the other.

Theorem 3.7. *Let S be a Sierpinski curve, and let A_1, A_2 be two arcs completely contained in the set of buried points B_S. Then there exists a homeomorphism $\widehat{H} : \mathbb{C} \to \mathbb{C}$ such that $\widehat{H}(A_1) = A_2$ and $\widehat{H}(S) = S$.*

To prove the theorem we need the following lemma.

Lemma 3.8. *Let S be a Sierpinski curve and let C_0 represent the outer boundary of S. Suppose A is an arc with $A \subset B_S$. Then there exists an arc T with endpoints $E \subset C_0$ such that $A \subset T \setminus E \subset B_S$.*

Proof. To find such an arc T let us consider the decomposition space $\widetilde{S}$ as defined in Proposition 3.6. Since $\widetilde{S}$ is homeomorphic to $\overline{\mathbb{D}}$, we can assume without loss of generality that $\widetilde{S} = \overline{\mathbb{D}}$. Let $\pi : S \to \overline{\mathbb{D}}$ be the quotient map induced by this decomposition, and set $R_S = S \setminus (C_0 \cup B_S)$. For any subset V of S, we will denote $\pi(V)$ by $\widetilde{V}$.

Notice we have that $(\widetilde{B}_S \cup \widetilde{C}_0) \cap \widetilde{R}_S = \emptyset$. This implies that $\widetilde{C}_0 = \partial\mathbb{D}$ and $\widetilde{A}$ is an arc in $\mathbb{D}$. It is sufficient to find an arc $\widetilde{T} \subset \overline{\mathbb{D}}$ with endpoints $\widetilde{E} \subset \partial\mathbb{D}$ such that $\widetilde{A} \subset \widetilde{T} \setminus \widetilde{E} \subset \widetilde{B}_S$, because $\pi^{-1}(\widetilde{T})$ is the desired arc.

Observe that since $\mathbb{C}$ is separable, $\widetilde{R}_S$ is countable. Next, consider an uncountable collection $\mathcal{L}$ of arcs in $\overline{\mathbb{D}}$ with endpoints in $\partial\mathbb{D}$ and having the following properties.

For every $l_1, l_2 \in \mathcal{L}$:

(i) $l_1 \cap l_2 = \widetilde{A}$,
(ii) if e_1 are the endpoints of l_1 then $l_1 \setminus e_1 \subset \mathbb{D}$.

Clearly such a collection exists. (To see this better, since any two arcs in $\mathbb{D}$ are equivalent under a homeomorphism of $\overline{\mathbb{D}}$, we may assume without loss of generality that $\widetilde{A}$ is a horizontal line segment.) Given such a collection, since $\widetilde{R}_S$ is countable, we can find an arc $\widetilde{T} \in \mathcal{L}$ with $\widetilde{T} \cap \widetilde{R}_S = \emptyset$. □

Proof. To begin the proof of Theorem 3.7, let S be a Sierpinski curve with arcs $A_1, A_2 \subset B_S$. Let C_0 be the outer boundary of S, and let G_0 be the region in $\mathbb{C}$ bounded by C_0. For A_1 let T_1 be the arc found in Lemma 3.8 and let E_1 be its endpoints. Likewise, define T_2 as the arc corresponding with A_2. Let $h: T_1 \to T_2$ be a homeomorphism with $h(A_1) = A_2$. We will first show that h can be extended to a homeomorphism $H: S \to S$ such that $H(C_0) = C_0$.

Observe that since C_0 is a simple closed curve, $C_0 \setminus T_1 = C_0 \setminus E_1$ is made of two components K_1, K_1'. Also $\overline{K_1}$ and $\overline{K_1'}$ are arcs with $\overline{K_1} \cap \overline{K_1'} = E_1$. Defining $C_1 = T_1 \cup K_1$ gives us a simple closed curve satisfying the properties needed in Proposition 3.4. So if we let G_1 be the region bounded by C_1, we have that $X_1 = \overline{G_1} \cap S$ is a Sierpinski curve with outer boundary C_1. We can likewise define $P_1 = T_1 \cup K_1'$ which bounds the region $H_1 = G_0 \setminus \overline{G_1})$ such that $Y_1 = \overline{H_1} \cap S$ is a Sierpinski curve with outer boundary P_1. Notice that by our construction $X_1 \cup Y_1 = S$ and $X_1 \cap Y_1 = C_1 \cap P_1 = T_1$.

In the same manner use T_2 to define the Sierpinski curves X_2 and Y_2 with the respective outer boundaries C_2 and P_2. We can then extend $h: T_1 \to T_2$ to homeomorphisms $f: C_1 \to C_2$ and $g: P_1 \to P_2$. By Proposition 3.3, we can extend f and g to homeomorphisms $F: X_1 \to X_2$ and $G: Y_1 \to Y_2$ respectively. Since $X_1 \cap Y_1 = T_1$ and $G|_{T_1} = h = F|_{T_1}$, we can define the homeomorphism $H: S \to S$ by the pasting lemma. From the construction of H we have that $H(C_0) = C_0$.

It is well known that a homeomorphism of a simple closed curve can be extended to a homeomorphism of $\mathbb{C}$. Hence, since $H(C_0) = C_0$ we can use the pasting lemma to extend H to a homeomorphism of $D_\infty \cup S$. Now by [7], if B is the boundary of a complementary domain, then $H(B)$ must also be the boundary of some complementary domain. Thus, since $H(C_0) = C_0$, we have that if B is the boundary of a bounded complementary domain D_1, then $H(B)$ is the boundary of some bounded complementary domain D_2. Clearly, we can extend H to a homeomorphism from $S \cup D_1$ to $S \cup D_2$. Doing this for every bounded complementary domain, it follows by the pasting lemma, and the fact that the bounded complementary domains of S have diameters going to 0, that we can extend H to a homeomorphism $\widehat{H}: \mathbb{C} \to \mathbb{C}$. □

We will use one more property of a Sierpinski curve. This property is proved "along the way" in [18].

Proposition 3.9. *Let B be the boundary of a complementary domain of the Sierpinski curve S. Then for every $\varepsilon > 0$ there exist a simple closed curve $Q \subset B_S$ with Hausdorff distance $< \varepsilon$ from B and such that Q encircles B.*

3.2. Example

To construct the example we will use the Knaster continuum K [8, Section 5] as our indecomposable subcontinuum. The Knaster continuum has the property that every proper subcontinuum $X \subset K$ is an arc or a point. This, together with a result by Krasinkiewicz [8, Property 3.4], gives us the following proposition.

Proposition 3.10. *Let $Y \subset \mathbb{C}$ be homeomorphic to the Knaster continuum. Then there exists an arc $A \subset Y$ such that any arc $X \subset \mathbb{C}$ with $X \cap A \neq \emptyset$ and $X \not\subset Y$, has the property that $X \cap (Y \setminus A) \neq \emptyset$.*

Now let us turn our focus towards the Julia set for the complex rational map $f(z) = z^2 - \frac{1}{16z^2}$. Let J denote the Julia set for this function. As shown in [1], J is a Sierpinski curve with $\mathbb{C}_\infty \setminus J$ being the basin of attraction of ∞. Let us consider one of the critical points ω of f. We have that ω lies in a complementary domain of J that is mapped two-to-one (except for the critical point) onto its image by f [1]. Hence by Proposition 3.9 and by the holomorphic properties of f we can find a simple closed curve $C \subset B_J$ such that f is two-to-one on C with $f(C)$ also being a simple closed curve. It follows that there is an arc $A \subset C$ such that f is one-to-one on A, except for its endpoints, and $f(A) = f(C)$.

Since Sierpinski curves are universal, there exists a subcontinuum $Y_0 \subset J$ that is homeomorphic to the Knaster continuum. Let $A_0 \subset Y_0$ be an arc as in Proposition 3.10. It follows that $A_0 \subset B_J$, the buried points of J. By Theorem 3.7 we can find a homeomorphism $\widehat{H}: \mathbb{C} \to \mathbb{C}$ such that $\widehat{H}(A_0) = A$ and $\widehat{H}(J) = J$. Define $Y = \widehat{H}(Y_0)$ and observe that Y is homeomorphic to the Knaster continuum. In fact, since $\widehat{H}$ is a homeomorphism of $\mathbb{C}$, we also have that $\widehat{H}(A_0) = A$ has the same property as the arc in Proposition 3.10 with respect to Y.

Recall that the simple closed curve C (found in the previous paragraph) contains A. Also, f is a locally a homeomorphism around noncritical points. Since the critical points are all in the Fatou set, this implies that $f(A)$ (while not an arc) has the same property as the arc in Proposition 3.10. For any arc $X \not\subset f(Y)$, if $X \cap f(A) \neq \emptyset$, then $X \cap (f(Y) \setminus f(A)) \neq \emptyset$. Thus, since each component of $\mathbb{C} \setminus f(A)$ is a domain with its boundary $f(A)$ being a simple closed curve, each component of $\mathbb{C} \setminus f(A)$ intersects $f(Y)$.

Let U, V be the components of $\mathbb{C} \setminus f(A)$. Then by the Boundary Bumping Theorem [12, p. 71], we have that $Z = \overline{U \cap f(Y)} \cup f(A)$ is a continuum. Since $A \subset Y$, $Z \subset f(Y)$. Furthermore $f(Y) \setminus Z = V \cap f(Y) \neq \emptyset$, so $Z \neq f(Y)$. Likewise we can define the subcontinuum $W = \overline{V \cap f(Y)} \cup f(A) \subset f(Y)$, and $W \neq f(Y)$. Clearly $W \cup Z = f(Y)$ and hence $f(Y)$ is decomposable.

3.3. Questions

We end with two questions.

Question 3.11. Is there a rational function (polynomial) whose Julia set is an indecomposable continuum?

Question 3.12. Is there a rational function (polynomial) with Julia set J nowhere dense in $\mathbb{C}_\infty$ such that J properly contains an invariant indecomposable subcontinuum?

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